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1996 J. Phys. A: Math. Gen. 29 2671

(http://iopscience.iop.org/0305-4470/29/11/008)

J. Phys. A: Math. Gen. **29** (1996) 2671–2685. Printed in the UK

Effect of packing method on the randomness of disc packings

Z P Zhang†, A B Yu† and R B S Oakeshott‡

† School of Materials Science and Engineering, The University of New South Wales, Sydney, NSW 2052, Australia

‡ ERC Tigress, PSTI Innovation Centre, Bridge of Don, Aberdeen AB23 8GX, UK

Received 8 August 1995, in final form 24 November 1995

Abstract. The randomness of disc packings, generated by random sequential adsorption (RSA), random packing under gravity (RPG) and Mason packing (MP) which gives a packing density close to that of the RSA packing, has been analysed, based on the Delaunay tessellation, and is evaluated at two levels, i.e. the randomness at individual subunit level which relates to the construction of a triangle from a given edge length distribution and the randomness at network level which relates to the connection between triangles from a given triangle frequency distribution. The Delaunay tessellation itself is also analysed and its almost perfect randomness at the two levels is demonstrated, which verifies the proposed approach and provides a random reference system for the present analysis. It is found that (i) the construction of a triangle subunit is not random for the RSA, MP and RPG packings, with the degree of randomness decreasing from the RSA to MP and then to RPG packing; (ii) the connection of triangular subunits in the network is almost perfectly random for the RSA packing, acceptable for the MP packing and not good for the RPG packing. Packing method is an important factor governing the randomness of disc packings.

Nomenclature

a, i, j, k	integers
d	diameter of discs
E_i	fractional number of the ith type of edge
$E_{\Delta ik}$	the number of the kth type of edge in an ith type of triangle
l	edge length
l_{min}	the shortest length of edges
m	the number of the types of triangles
n	the number of the types of edges
N_T	the total number of triangles
N_{ij}	fractional number of the connections between the ith and jth types of triangles
N_{ij}^{c}	the calculated value of N_{ij} by equation (5)
N_{ij}^{m}	the measured value of N_{ij}
SSR	sum of squared residuals
T_i	fractional number of the ith type of triangles
$T_{\Delta ijk}$	fractional number of the triangles constructed by the ith, jth and kth types of edges
$T_{\Delta ijk}^{\mathrm{c}}$	calculated $T_{\Delta ijk}$
$T_{\Delta ijk}^{\mathrm{m}}$	measured $T_{\Delta ijk}$
ε_{ij}	absolute difference between N_{ij}^{c} and N_{ij}^{m}
$\bar{\varepsilon}$	average value of ε_{ij}.

0305-4470/96/112671+15$19.50

1. Introduction

Random packing of spheres has been studied quite extensively for models to represent the structure of liquids and glasses [1, 2] or to investigate such phenomena as fluid flow [3–5], electrical conductivity [6–8] and force transmission [9] in packed particles. This subject has many industrial applications as listed by German [10]. However, the randomness of the structure of particle packings, which is not completely random as it implies, has not been tested systematically. Proper quantification of the randomness of particle packing is important to the successful application of a 'random' model to practical problem solving. In fact, the evaluation of the 'degree of randomness' has been known as one of the main problems in the area of particle or disc packing [11].

Two methods are widely used in the structural analysis of particle or disc packing, namely the Voronoi and Delaunay tessellations [12, 13]. Consequently, the structural analysis can be considered from two aspects: the properties of individual Voronoi or Delaunay subunits and the properties of these combined or connected subunits. The geometrical properties of the Voronoi subunit such as the side number distribution, edge length distribution etc, have been studied extensively, particularly for disc (2D) packings [14, 15]. However, it seems that to date no detailed analysis has been carried out on the connection of the subunits. This is probably because of the complicated topology of the Voronoi tessellation as compared to that of the Delaunay tessellation. With this realization, Mellor investigated the randomness of Finney's sphere (3D) packing applying the Delaunay tessellation [16]. Two levels of randomness are identified and analysed, which are, respectively, related to the construction of individual subunits, i.e. tetrahedra for the 3D situation, and the connection between these subunits. He found that for Finney's packing, the construction of tetrahedra is not randomly combined from the measured edge length distribution, but the connection between tetrahedra are reasonably random. His work appears to be the first in the literature to discuss the randomness in such a quantitative and detailed way. As pointed out by Finney [17], the work in this direction deserves more attention in future studies.

In this paper we will extend Mellor's work by investigating the effect of packing method on the randomness of disc packing. The method for analysing the randomness for disc packing will first be rationalized and generalized. Then, the randomness of disc packings obtained by various computer simulation algorithms will be analysed.

2. Method for analysis

According to the Delaunay tessellation, a packing of discs can be divided into a certain type of triangular network obtained by connecting the centres of adjacent discs. The randomness of the packing can then be analysed at two levels. The first is the randomness at individual subunit level which relates to the construction of a triangle from a given edge length distribution. The second is the randomness at network level which relates to the connection between triangles from a given triangle frequency distribution. In the following, the general definition of the randomness at the two levels will be provided.

For convenience, an edge length frequency distribution is first discretized into a number of types of edges. Let n represent the number of the types of the edges and E_i the fractional number, i.e. the frequency of the ith type of edge with the following normalization condition:

$$\sum_{i=1}^{n} E_i = 1\,. \tag{1}$$

A triangle consists of three edges. Let $T_{\Delta ijk}$ be the fractional number of triangles constructed by the i, j and kth types of edges. $T_{\Delta ijk}$ is obviously dependent on E_i. In fact, if triangles are constructed as a result of the random combination of any three types of edges, $T_{\Delta ijk}$ should theoretically be related to E_i and given as the respective terms of the expansion

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} T_{\Delta ijk} = \left(\sum_{i=1}^{n} E_i\right)^3 \tag{2a}$$

or

$$T_{\Delta ijk} = \begin{cases} E_i^3 & (i = j = k) \\ 3E_i^2 E_k & (i = j \neq k) \\ 3E_i E_j^2 & (i \neq j = k) \\ 6E_i E_j E_k & (i \neq j \neq k)\,. \end{cases} \tag{2b}$$

It seems that the randomness at the individual subunit level can be analysed, based on this equation. However, since this equation does not take into account the geometrical constraint in forming a packing, this analysis is only applicable to dense packing, as originally proposed by Mellor [16], and cannot be used generally. This can be understood from the fact that the construction of a triangle must satisfy the geometric constraint that the sum of the lengths of two edges is greater than the length of the other. The random combination of three edges from a given edge length distribution may not always give a triangle that physically exists if the edge length varies in a large range. In this case, the discussion of the randomness at individual subunit level is not meaningful if it is based on equation (2) without modification. To overcome this deficiency, the maximum entropy method as used by other investigators in the analysis of the Voronoi structure [18] was employed in this study. For the present problem, the entropy S is defined as

$$S = -\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} T_{\Delta ijk} \ln T_{\Delta ijk}\,. \tag{3a}$$

The most random distribution of $T_{\Delta ijk}$ can then be obtained by maximizing the entropy, subject to the constraints

$$3E_a = T_{\Delta aaa} + \sum_{i=1}^{a} T_{\Delta iaa} + \sum_{k=a+1}^{n} T_{\Delta aak} + \sum_{j=a+1}^{n}\sum_{k=j}^{n} T_{\Delta ajk} + \sum_{i=1}^{a}\sum_{k=a+1}^{n} T_{\Delta iak} + \sum_{i=1}^{a}\sum_{j=i}^{a} T_{\Delta ija} \quad (a = 1, 2, \ldots, n) \tag{3b}$$

and

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} T_{\Delta ijk} = 1\,. \tag{3c}$$

Equation (3*b*) means that the calculated triangular distribution will always provide the same edge length distribution as that given and equation (3*c*) is the normalization condition. By definition, $T_{\Delta ijk}$ should be greater than or equal to zero. $T_{\Delta ijk} = 0$ should occur when the combination of three edges gives a triangle which does not satisfy the above geometrical constraint. The types of triangles which do not exist physically can be readily determined for a given (discretized) edge length distribution. In this way, in the evaluation of $T_{\Delta ijk}$, not only the edge length distribution but also the geometric constraint have been taken into account. The randomness at individual subunit level can then be analysed, based on equation (3), as used in this work. The construction of triangles is said to be of perfect randomness at individual subunit level if $T_{\Delta ijk}$ can be evaluated by this method.

Now we will consider the randomness at network level. Different combinations of edges will give different triangles. The number of the types of triangles m should be linked with n, as given by $m = \binom{n+2}{3}$. Let T_i be the number fraction of the ith type of triangle with the following normalization condition:

$$\sum_{i=1}^{m} T_i = 1\,. \tag{4}$$

Then the number of connections between triangle i and triangle j will depend on T_i and T_j and on whether there is any type of common edge in the two types of triangles. Assume that a packing is composed of N_T triangles (N_T can be determined from the number of discs according to Euler's theorem). If $E_{\Delta ik}$ is the number of the kth type of edge in the ith type of triangle, then $N_T T_i E_{\Delta ik}$ will be the total number of the kth type of edges in the ith type of triangle. If the connection between the ith and jth types of triangles is random, the contribution of the kth type of edge to this connection should be equal to $(N_T T_i E_{\Delta ik}) \times (N_T T_j E_{\Delta jk}) / \sum_{i=1}^{m} (N_T T_i E_{\Delta ik})$. Note that k varies from 1 to n, i.e. the connection between the ith and jth types of triangles can be made by all types of edges, and the total number of connections between triangles is $3N_T$. The fractional number of the connection between the ith and jth types of triangles N_{ij} should then be given by the following equation:

$$N_{ij} = \frac{1}{3} \sum_{k=1}^{n} \left(\frac{T_i E_{\Delta ik} T_j E_{\Delta jk}}{\sum_{i=1}^{m} T_i E_{\Delta ik}} \right). \tag{5}$$

The present analysis of the randomness at network level will be based on this equation. In particular, the connection between triangles from a given triangle frequency distribution is said to be perfectly random if the connection between any two types of triangles can be evaluated by equation (5).

As will be discussed later, the use of the above approach allows one to discuss the randomness of a 2D disc packing in a quantitative manner. This approach can also be extended to a 3D sphere packing. From this point of view, the analysis of Mellor [16] is concerned with the simplest case because only two types of edges were employed in his analysis. Furthermore, Mellor's definition of the randomness at individual level, like that related to equation (2), is only applicable to dense packing with a narrow edge length distribution and cannot be used as a general method. As will be discussed in section 4.1, this deficiency can be satisfactorily overcome by the above proposed entropy approach.

3. Computer simulation of disc packing

For convenience, the structural information necessary for the present analysis was generated by computer simulation. Three simulation algorithms were employed, which represent different packing methods and hence structures. These methods are the random sequential adsorption (RSA) [19], the random packing under gravity (RPG) [20, 21] and the Mason packing (MP) [22]. The simulation algorithms have been detailed in a study of the pore structure of disc packing [23], or in the literature [19–22], and will therefore not be repeated here.

The packing density obtained in our simulation is 0.543 for the RSA and 0.840 for the RPG packing. These values agree well with those obtained by other investigators [19, 20, 24]. For the MP algorithm, packing density varies theoretically from zero to 0.906, depending on the growth state of discs [22]. However, in order to make it comparable with the RSA packing, only the packing with packing density equal to 0.536 was studied in this work. Moreover, a pure Delaunay tessellation (DT) constructed by random points was also investigated in detail in order to quantify the possible effect of the standard technique to obtain a Delaunay tessellation on the randomness. This resulting Delaunay tessellation could be regarded as an extreme RSA or MP packing when the disc diameter is equal to zero.

Therefore, in total, four simulation algorithms were used in this work. Figure 1 shows the packing structures and their Delaunay tessellations obtained by these algorithms. The structures were analysed numerically in detail to obtain the necessary statistical information about edge length distribution, triangle frequency distribution, connection between triangles, etc, for the present analysis which was aimed at elucidating the effect of packing method on the randomness of disc packings.

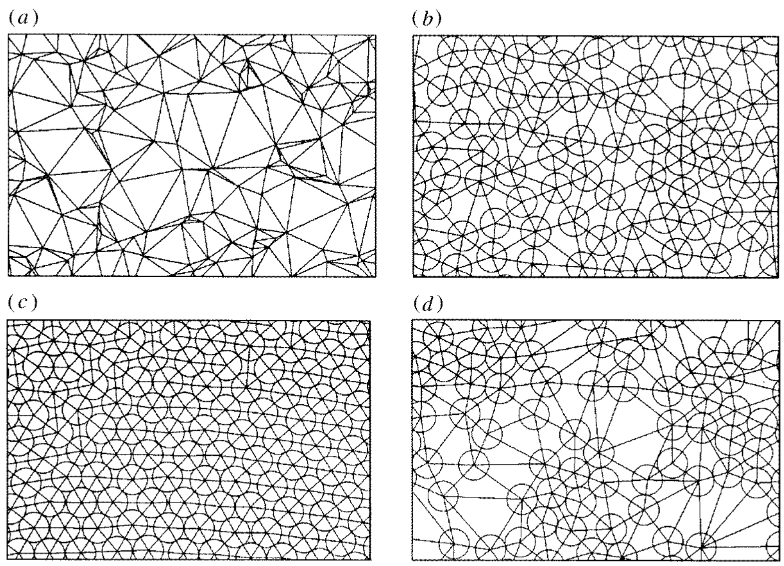

Figure 1. Typical packing structures and Delaunay subunits for (*a*) DT, (*b*) RSA, (*c*) RPG and (*d*) MP packings.

4. Results and discussion

4.1. The randomness at individual subunit level

Figure 2 shows the edge length distributions corresponding, respectively, to the DT, RSA, RPG and MP packings. To examine the randomness at individual subunit level, these distribution functions were discretized and E_i determined. By maximizing the entropy defined by equation (3), the triangle frequency distribution $T_{\Delta ijk}$ could be determined, which were then compared with those measured. If the construction of the triangles from the edge length distribution for a packing is random, the measured and calculated $T_{\Delta ijk}$ should match each other well. Otherwise, it is not random. The difference between the measured and calculated $T_{\Delta ijk}$ is obviously a measure of the degree of randomness.

To evaluate the randomness at individual subunit level, four types of edges were given, giving 20 types of triangles. For the DT and RSA packings, the edge intervals in terms of dimensionless edge (refer to figure 2) are equally distributed from the minimum (equal to unity) to maximum (equal to 370.8256 for the DT and 1.9977 for the RSA packing). This

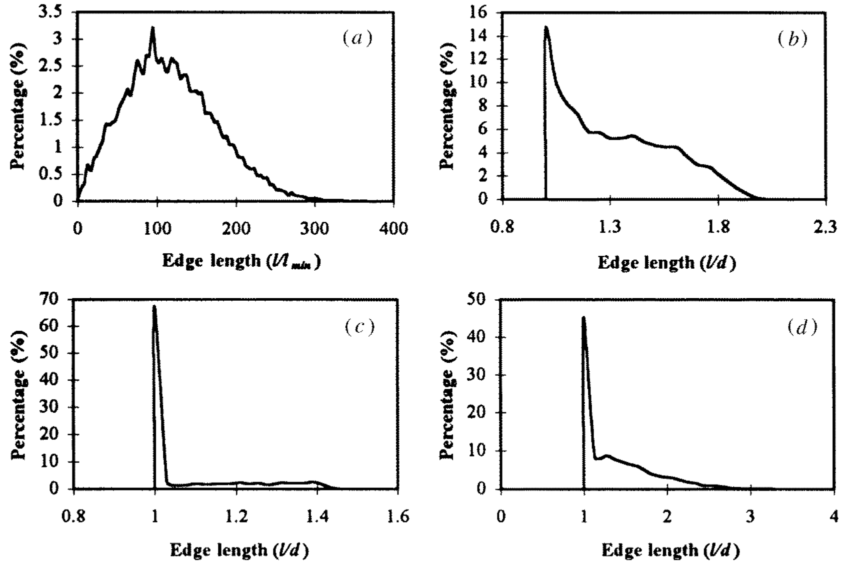

Figure 2. Edge length distributions for (*a*) DT, (*b*) RSA, (*c*) RPG and (*d*) MP packings.

treatment was also applied to the RPG and MP packings for the largest three edge intervals, with the maximums, respectively, equal to 1.4884 for the RPG and 3.3837 for MP packing. However, the smallest edge interval was set to be from unity to 1.001 for the RPG and to 1.0001 for MP packing. The reason for this treatment is that the length of the majority of edges for the RPG and MP packings is distributed within a narrow range (figures 2(*c*) and (*d*)). It was considered, after some trials, that this discretization could lead to more meaningful results. This treatment was also adopted in the analysis of the randomness at network level.

Figure 3 shows the comparison between the calculated and measured triangle frequency distributions for these packings. It can be observed from figure 3(*a*) that for the DT packing, the calculated $T_{\Delta ijk}$ are in good agreement with the measured $T_{\Delta ijk}$, suggesting that the construction of triangles for this packing is almost of perfect randomness. This result is expected since the DT packing is essentially a random system without any packing constraints. However, as observed from other figures (in figure 3), the calculated $T_{\Delta ijk}$ are significantly different from the measured $T_{\Delta ijk}$. This suggests that for the RSA, RPG and MP packings, the construction of the individual triangle is not random. To be more quantitative, we used the sum of squared residuals, i.e. $\mathrm{SSR} = \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n}(T^{\mathrm{c}}_{\Delta ijk} - T^{\mathrm{m}}_{\Delta ijk})^2$ in this analysis. Figure 4 shows the results, which indicate that the degree of randomness varies for the four packings. In summary, the degree of randomness at an individual subunit level is excellent for the DT packing but gets worse from the RSA to the MP and finally to the RPG packing.

The degree of randomness is considered to be related to the packing constraints imposed on each type of packing. As mentioned above, there is no packing constraint in the DT packing. It is therefore expected that it should give the highest degree of randomness. For the RSA packing, the centre of each disc is chosen at random and the only determinant constraint is the steric exclusion [24]. However, for the MP packing, in addition to the constraint of the steric exclusion, the coordinates of each disc are affected by contacting neighbours during the growth process of discs [22]. As a consequence, the MP packing

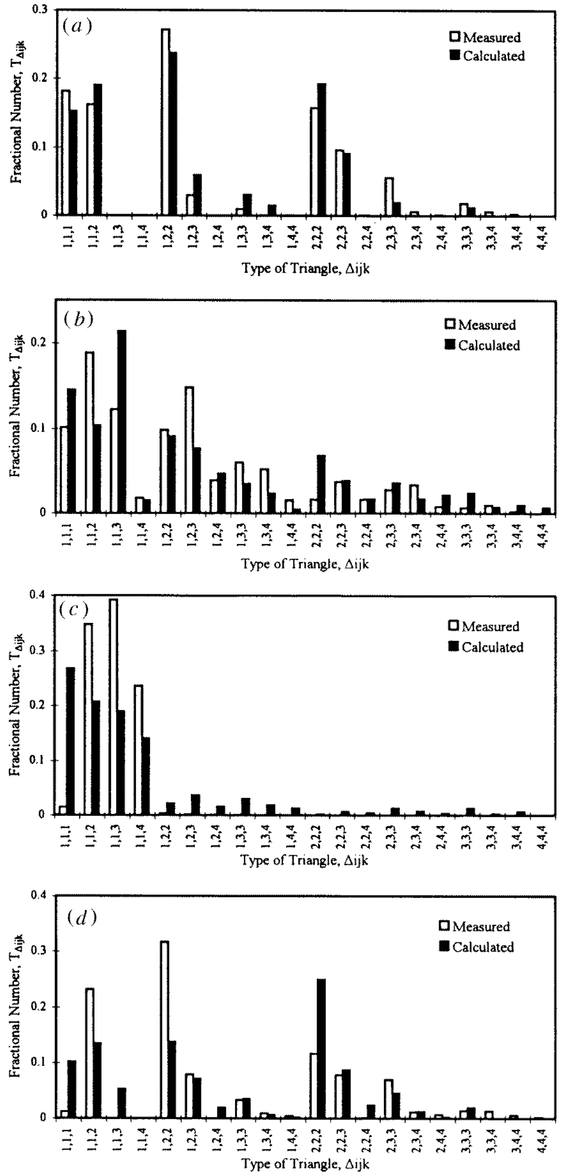

Figure 3. Comparison between the measured and calculated triangle frequency distributions for (*a*) DT, (*b*) RSA, (*c*) RPG and (*d*) MP packings.

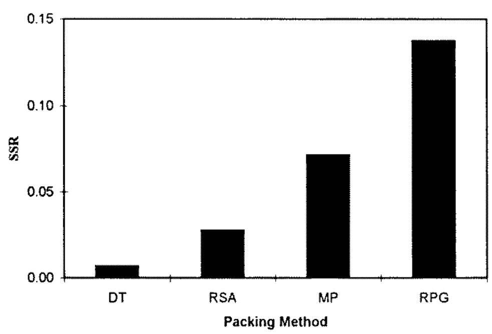

Figure 4. Effects of packing methods on the sum of squared residuals in evaluating $T_{\Delta ijk}$.

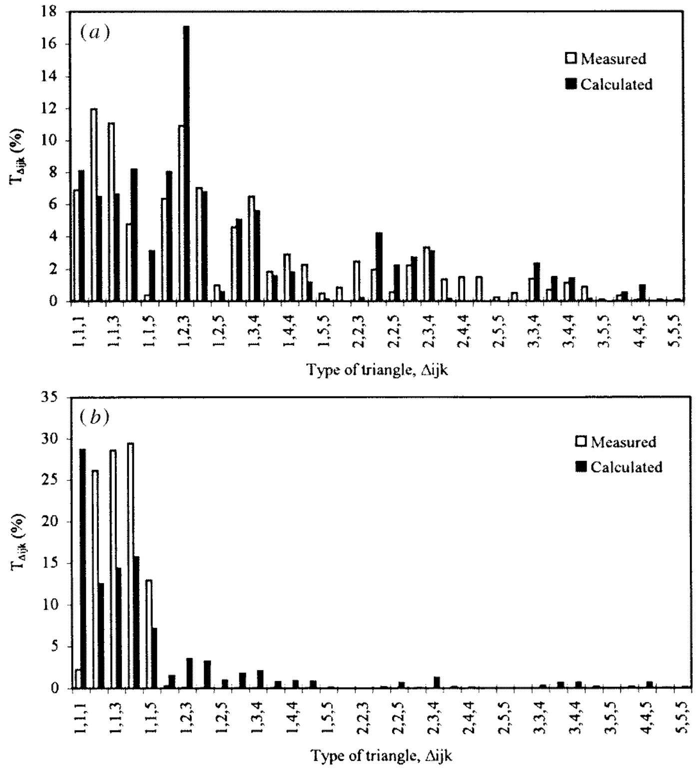

Figure 5. Comparison between the calculated and measured triangle frequency distributions for (*a*) RSA and (*b*) RPG packings (based on equation (2)).

gives a lower degree of randomness than the RSA packing. As for the RPG packing, the effect of gravity implies that a disc must be supported by two discs underneath and it will also provide such support to at least one disc. Consequently, the majority of measured triangles in the RPG packing are isosceles triangles (see figure 3(*c*)). Because of this strong packing constraint, the measured $T_{\Delta ijk}$ are quite different from the calculated $T_{\Delta ijk}$, giving the lowest degree of randomness. The above consideration is also applied to the analysis of the randomness at the network level. In fact, it is considered that the packing method affects the degree of randomness of disc packing mainly through its imposed packing constraints.

It should be emphasized that in this work we only considered the MP packing with a packing density close to that of the RSA packing. The randomness of the MP packing varies with packing density, i.e. the growth process of discs. The above comparison between the MP and RSA packings should therefore not be generalized. Further study is probably necessary in order to fully establish the relationship between the randomness of the MP packing and packing density.

As mentioned above, $T_{\Delta ijk}$ may also be evaluated by equation (2) and the calculated $T_{\Delta ijk}$ can also be used for the analysis of randomness. In fact, such analysis was carried out for the RSA and RPG packings with narrow edge length distribution. Figure 5 shows the results, obtained when the edge length distribution was discretized into five intervals. These intervals are equally distributed from minimum to maximum for the RSA packing. For the RPG packing, for the reason mentioned above, the equally distributed intervals were only made for the large four intervals and the smallest interval was taken from unity to 1.001. It can be seen from figure 5 that the calculated $T_{\Delta ijk}$ are different from the measured $T_{\Delta ijk}$, so that the construction of triangles for the two packings is not random. This remark is obviously in agreement with that obtained by the maximum entropy method. Note that the results in figure 5 are not necessarily the same as those in figure 3 as they are obtained using different approaches. It was also observed that for the RSA packing, if the number of intervals is smaller than five, the agreement between the measured and calculated $T_{\Delta ijk}$ by equation (2) can be significantly improved. A large number of intervals would lead to a more conclusive result as long as the number of discs involved in the analysis is large enough.

However, it should be pointed out that the analysis based on equation (2) is not applicable to the DT and MP packings. As seen from figure 2, the length of edges for the two packings varies within such a wide range that the random combinations of three edges may not always lead to triangles which are realistic. In this case, the use of equation (2) would definitely lead to a conclusion that the construction of triangles for either of the two packings is not random. This is obviously unreasonable, if not absurd. This is because equation (2) has not taken into account the geometrical constraint as discussed above. Obviously, the analysis based on the maximum entropy method can overcome this problem and can be used generally, though it involves greater numerical effort in problem solving.

4.2. Randomness of the Delaunay tessellation at network level

The randomness at the network level can be accessed using a similar method to the above discussion. Here the measured triangle frequency distribution is used as the prescribed information to calculate the fractional number of connections between the ith and jth types of triangles N_{ij} according to equation (5). This calculated N_{ij}^{c} can then be compared with the measured N_{ij}^{m} to evaluate the randomness at the network level.

For the packing of an infinitely large number of discs, a large n will give a large m and hence an accurate classification of the types of triangles, which in turn leads to a more precise discussion of the randomness of the packing structure. However, because of the limited computing capacity and time, the number of discs generated in a packing is always limited. This is particularly true when a collective simulation algorithm, e.g. the MP algorithm used here, is employed. In this case, the result of randomness is strongly affected by the value of n, i.e. the number of edge intervals. This effect can be found from the results at both the individual and network levels but is much more significant for the latter, because the types of connection increase sharply with the number of intervals. To provide a sound base for the analysis at network level this effect was studied in detail with reference to the DT packing, i.e. the pure Delaunay tessellation, as described earlier.

As mentioned above, the difference between N_{ij}^{c} and N_{ij}^{m} and, in particular, $\varepsilon_{ij} = |N_{ij}^{\mathrm{c}} - N_{ij}^{\mathrm{m}}|$, may be used as a measure of the degree of randomness. For a given number of edge intervals n, ε_{ij} varies with i and $j (\leqslant i \leqslant j \leqslant n)$. For simplicity, their overall average, $\bar{\varepsilon}$, was used in the present analysis. Figure 6 shows the variation of this $\bar{\varepsilon}$ with the number of discs (points for the DT packing) for different edge intervals. It appears that the number of discs should be larger than 6000 in order to obtain a stable $\bar{\varepsilon}$ and hence a statistically meaningful result. Moreover, since this stable $\bar{\varepsilon}$ also varies with the number of intervals, to obtain the comparable results from different simulation algorithms, the number of edge intervals should be the same. Therefore, the number of edge intervals used in this work is constant and equal to 4 while the number of discs is greater than 6000. This gives 20 types of triangles and 400 types of connections between triangles.

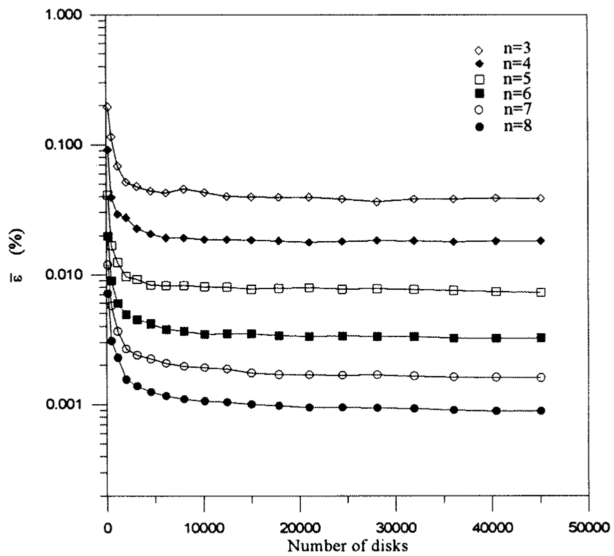

Figure 6. Variation of $\bar{\varepsilon}$ with the number of discs for different edge intervals for the DT packing.

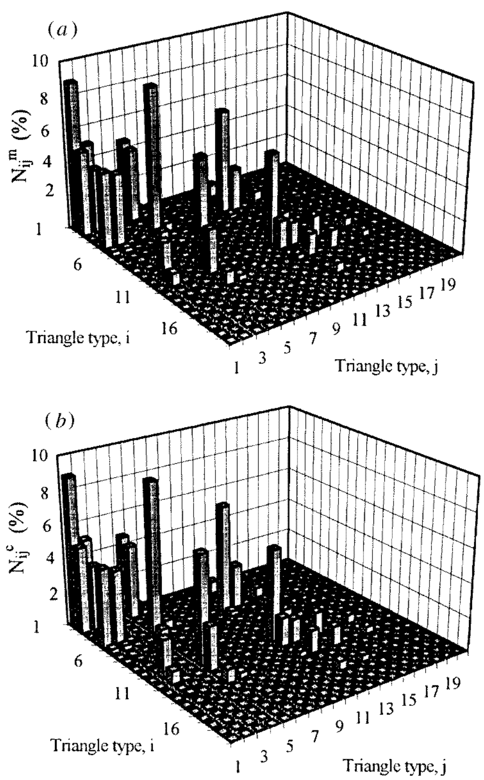

Figure 7. Comparison between (*a*) measured and (*b*) calculated N_{ij} for the DT packing.

Figure 7 shows the comparison between the measured and calculated N_{ij} for the DT packing for the 20 types of triangles. It appears that the calculated N_{ij}^{c} match the measured N_{ij}^{m} well. This good agreement can be seen from the direct comparison between N_{ij}^{m} and N_{ij}^{c} in figure 8. These results indicate that the Delaunay tessellation, as a subdivision method

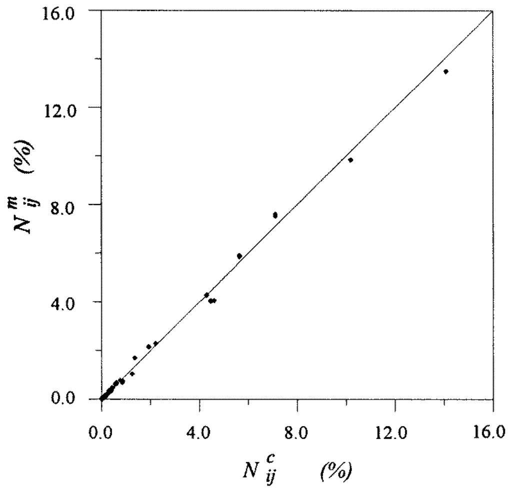

Figure 8. Direct comparison between N_{ij}^{m} and N_{ij}^{c} for the DT packing.

used to configurate a disc packing into a triangular network, would provide a very high degree of randomness at the network level which is expected to be very close to that of perfect randomness. The results also suggest that the Delaunay tessellation, when applied to other packings obtained by other simulation algorithms, should not affect the randomness at the network level much.

The above remarks stem from the analysis of the absolute difference between N_{ij}^{c} and N_{ij}^{m}. As N_{ij} is actually a frequency distribution, the difference between N_{ij}^{c} and N_{ij}^{m} can also be analysed in terms of a statistical test, e.g. the so-called χ^2-test which is widely used in testing the agreement between two distributions. In doing so, one would find that the measured and calculated N_{ij} are quite different. This results from the fact that by definition, χ^2 is proportional to the number of triangles, which is extremely large here (about 10^5). Consequently, a practically insignificant difference between N_{ij}^{c} and N_{ij}^{m} may result in a large χ^2 value and a failure in this statistical test. Mellor reached a similar conclusion in analysing Finney's 3D packing structure by this test [16].

However, it should be pointed out that the use of $\bar{\varepsilon}$ or another statistical index should not affect the final outcome since our aim was to evaluate the relative rather than absolute randomness for different types of packings. In this case, the Delaunay tessellation just provides a reference system which should have a maximum degree of randomness at network level. The analysis of the randomness of other packings can then be made by reference to the DT packing, as given below.

4.3. Randomness of the RSA, RPG and MP packings at network level

In this section we will consider the randomness of connections between triangles, i.e. the randomness at the network level, for the RSA, RPG and MP packings. Our analysis is mainly made using the same treatments as those for the DT packing. Figures 9 and 10 show the comparison between the measured and calculated N_{ij} for the RSA and RPG packings, respectively. For the RSA packing, the distribution function of N_{ij}, calculated by equation (5), matches those measured well. The good agreement between the calculated and measured N_{ij} can be seen from their direct comparison given in figure 11. In fact, the degree of randomness at the network level for the RSA packing is comparable with that for the DT packing. However, it appears that the RPG packing is a relatively worse random

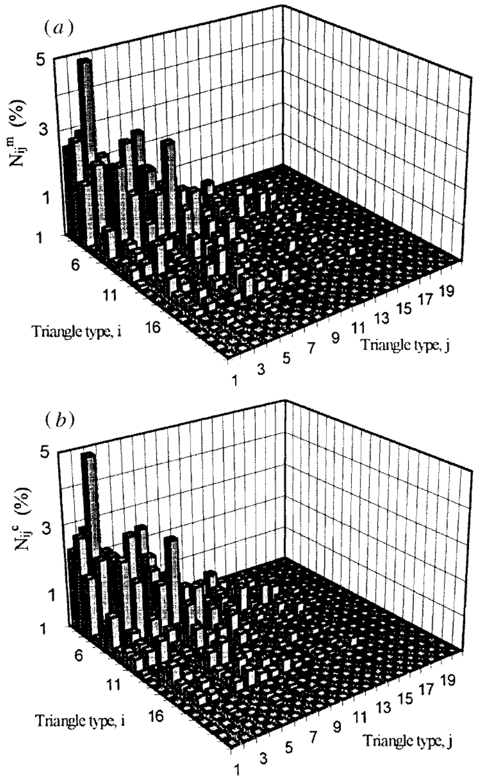

Figure 9. Comparison between (*a*) measured and (*b*) calculated N_{ij} for the RSA packing.

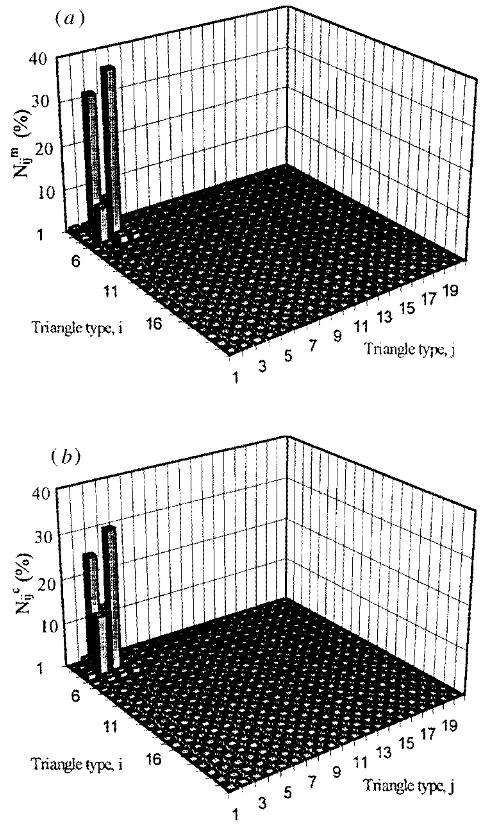

Figure 10. Comparison between (*a*) measured and (*b*) calculated N_{ij} for the RPG packing.

packing at this level, though there is still general agreement between the calculated and measured N_{ij}, as seen from figure 10.

Figure 12 plots the average error $\bar{\varepsilon}$ as a function of packing method. Obviously, the best randomness is obtained for the DT packing matched with the RSA packing, and the worst randomness corresponds to the RPG packing. The relatively low degree of randomness with the RPG packing is due to the need to meet the stability requirement imposed by the gravity, i.e. the packing constraints as discussed above.

Figure 13 shows the measured and calculated N_{ij} for the MP packing with a packing density of 0.536. Since the number of discs used in this work was 1200 and less than 6000, only two types of edges were classified, giving four types of triangles and 16 types of connection. The similarity between the calculated and measured results suggests that the packing structure of this particular MP packing is approximately random at the network level. Further analysis of the MP packing was also carried out by comparing its average $\bar{\varepsilon}$ to that of the RSA packing composed of the same number of discs. It was found that the $\bar{\varepsilon}$ value is 0.6% for the MP packing and 0.3% for the RSA packing, so that the RSA algorithm generates a packing of discs more random than the MP algorithm. Note that the MP algorithm can generate packings with packing density varying from infinitely small to 0.906. It is not clear how this will affect the randomness. As noted above, this will be studied in detail in the future. However, the present comparison between the MP and RSA

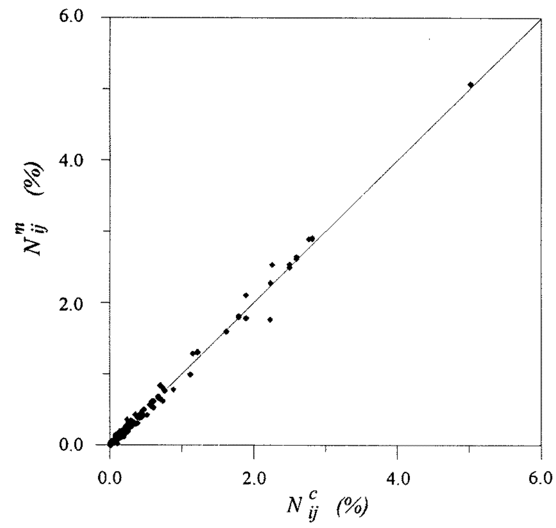

Figure 11. Direct comparison between N^{m}_{ij} and N^{c}_{ij} for the RSA packing.

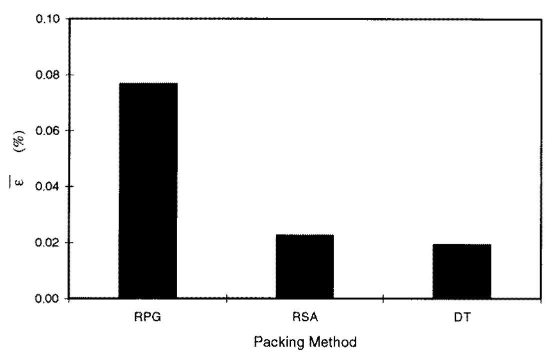

Figure 12. Plot of $\bar{\varepsilon}$ as a function of packing method.

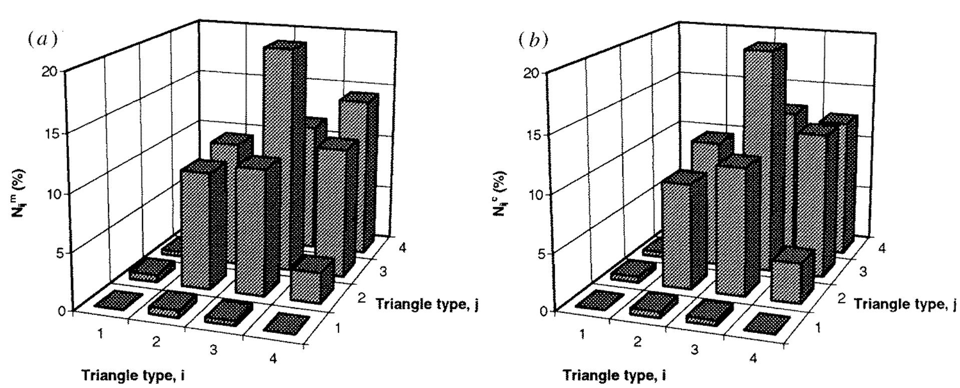

Figure 13. Comparison between (*a*) measured and (*b*) calculated N_{ij} for the MP packing.

packings clearly indicates that the packing density is not as dominant as the packing method in determining the randomness of a disc packing.

5. Conclusions

The randomness of disc packings, generated by different computer simulation algorithms such as the random sequential adsorption (RSA), the random packing under gravity (RPG) and the Mason packing (MP) which gives a packing density close to that of the RSA packing, has been analysed, based on the Delaunay tessellation. The degree of randomness is evaluated at two levels, i.e. the randomness at individual subunit level which relates to the construction of individual triangles from a given triangle frequency distribution and the randomness at network level which relates to the connection between triangles from a given triangle frequency distribution. The method of analysis, as originally used by Mellor [16], has been rationalized and generalized for multiple types of edges and triangles. In particular, the maximum entropy method has been employed in order to properly take into account the geometrical constraint in forming a packing. The Delaunay tessellation itself is analysed and its almost perfect randomness at the two levels is demonstrated, which verifies the proposed approach and provides a random reference system for the analysis of the other packings. It is found that (i) the construction of a triangle subunit is not random for the RSA, MP and RPG packings, with the degree of randomness decreasing from the RSA to MP and then to RPG packing; (ii) the connection of triangular subunits in the network is essentially random for the RSA packing, acceptable for the MP packing, and not good for the RPG packing. Different packing methods impose different packing constraints which in turn affect the degree of randomness. It is concluded that packing method is an important factor governing the randomness of disc packings.

Finally, we would like to point out that understanding the randomness at various levels is important in the modelling of the relationship between micro- and macrostructural properties for practical application. It is obvious that this modelling can be readily made if a packing is random at all levels. However, perfect random packing is rarely found in reality. Identification of the non-randomness at various levels may lead to the determination of the 'minimum' information required to properly characterize a particle/disc packing. A predictive model can therefore be developed on the basis of this minimum information. This would give a promising approach to the mathematical description of the packing of particles or discs.

References

[1] Bernal J D 1960 Geometry of the structure of monatomic liquids *Nature* **185** 68
[2] Bennett C H 1972 Serially deposited amorphous aggregates of hard spheres *J. Appl. Phys.* **43** 2727
[3] Bryant S L, King P R and Mellor D W 1993 Network model evaluation of permeability and spatial correlation in a real random sphere packing *Transport in Porous Media* **11** 53–70
[4] Guyon E, Oger L and Plona T J 1987 Transport properties in sintered porous media composed of two particle sizes *J. Phys. D: Appl. Phys.* **20** 1637–44
[5] Oger L, Guyon E and Wilkinson D 1987 Permeability variation due to spherical impurities in a disordered packing of equal spheres *Europhys. Lett.* **4** 301–5
[6] Davis R A and Deresiewicz H 1977 A discrete probabilistic model for mechanical response of a granular medium *Acta Mechanica* **27** 69
[7] Travers T, Ammi M, Bideau D, Gervois A, Messager J C and Troadec J P 1987 Uniaxial compression of 2D packing of cylinders. Effects of weak disorder *Europhys. Lett.* **4** 329–32
[8] Madhavrao L R and Rajagopalan R J 1989 Monte Carlo simulations for sintering of particle aggregates *J. Mater. Res.* **4** 1251
[9] Travers T, Bideau D, Gervois A, Troadec J P and Messager J C 1986 Uniaxial compression effects on 2D mixtures of 'hard' and 'soft' cylinders *J. Phys. A: Math. Gen.* **19** L1033–8
[10] German R M 1989 *Particle Packing Characteristics* Metal Powder Industries Federation, Princeton, NJ

[11] Weaire D and Rivier N 1984 Soap, cells and statistics—random patterns in two dimensions *Contemp. Phys.* **25** 59–99

[12] Hermann H, Wendrock H and Stoyan D 1989 Cell-area distribution of planar Voronoi mosaics *Metallograph* **23** 189–200

[13] Green P J and Sibson R 1978 Computing Dirichlet tessellations in the plane *Comput. J.* **21** 168–73

[14] Gervois A, Troadec J P and Lemaitre J 1992 Universal properties of Voronoi tessellations of dard discs *J. Phys. A: Math. Gen.* **25** 6169–77

[15] Lemaitre J, Gervois A, Troadec J P, Rivier N, Ammi M, Oger L and Bideau D 1993 Arrangement of cells in Voronoi tessellations of monosize packing of discs *Phil. Mag.* B **67** 347–62

[16] Mellor D W 1989 Random close packing (RCP) of equal spheres: structure and implications for use as a model porous medium *PhD Dissertation* Open University

[17] Finney J 1993 Local structure of disordered hard sphere packings *Disorder and Granular Media* ed D Bideau and A Hausen (Amsterdam: Elsevier) p 35

[18] Rivier N and Lissowski A 1982 On the correlation between sizes and shapes of cells in epithelial mosaics *J. Phys. A: Math. Gen.* **15** L143–8

[19] Hinrichsen E L, Feder J and Jøssang T 1986 Geometry of random sequential adsorption *J. Stat. Phys.* **44** 793

[20] Visscher W M and Bolsterli M 1972 Random packing of equal and unequal spheres in two and three dimensions *Nature* **239** 540

[21] Barker G C and Grimson M J 1989 Sequential random close packing of binary disc mixtures *J. Phys.: Condens. Matter* **1** 2779–89

[22] Mason G 1976 Computer simulation of hard disc packings of varying packing density *J. Colloid Interface Sci.* **56** 483

[23] Zhang Z P, Yu A B and Dodds J A 1995 Computer simulation of the packing of discs *5th Int. Conf. Bulk Materials Storage, Handling and Transportation (Newcastle, Australia)* vol 2, pp 331–4

[24] Berryman J G 1983 Random close packing of hard spheres and discs *Phys. Rev.* A **27** 1053